Small-amplitude 2D patterns with nontrivial symmetry in a simple nonlinear field model

2000 J. Phys. A: Math. Gen. 33 2469

(http://iopscience.iop.org/0305-4470/33/13/302)

J. Phys. A: Math. Gen. **33** (2000) 2469–2487. Printed in the UK PII: S0305-4470(00)01889-8

Small-amplitude 2D patterns with nontrivial symmetry in a simple nonlinear field model

V M Eleonsky, V G Korolev and N E Kulagin
Lukin Research Institute of Physical Problems, Zelenograd, Moscow, 103460, Russia
E-mail: korolev@nonlin.msk.ru

Received 17 February 1999, in final form 30 November 1999

Abstract. Quasiperiodic (QP) small-amplitude patterns are studied in a scalar field theory with quadratic nonlinearity. QP solutions of the class in interest are found as a projection of strictly periodic solutions of an associated 4D problem onto an 'irrationally oriented' 2D subspace. The periodic solutions of the 4D problem are constructed using a version of the method of asymptotic expansions. The analysis reveals complex patterns. In particular, there exists a one-parametric QP pattern with strict 12-fold symmetry, which contains infinitely many local patches with approximate 5-fold symmetry. In limit cases, the complex patterns transform into a simple pattern: a close pack of hexagonal cells. In certain resonance cases there exist patterns consisting of alternating pieces of close cell packs with either hexagonal or quadrangular symmetry. The relation between the 12-fold and 5-fold approximate symmetries is discussed.

1. Introduction

When one develops a theory that has to describe evolution processes in nonlinear media involving phenomena of self-organization, form creation, or 'order–chaos' transitions, there always arises the question: which is a variety of structures and forms admissible by one or another nonlinear field model? Even for simple models with spatially multidimensional nonlinear field distributions this question may be answered only after analysing the structure of stationary solutions.

What structures are encountered in nature and thus should be described by a theoretical model? In various nonlinear media one typically observes simple patterns, which are associated with strictly periodic nonlinear wave lattices with certain symmetry. Such patterns exist, for instance, in a vortex lattice in a stationary flow of an ideal liquid [1], in a lattice of optical filaments in nonlinear optical media [2, 3] etc. For corresponding models, it is necessary to investigate the bifurcations of solutions of the nonlinear field equation (when a small-amplitude wave lattice arises on the background of the uniform field distribution) and to study the evolution of such small-amplitude lattices as the characteristic amplitude (the norm of the solution) is varied.

But, together with analysing spatially multidimensional patterns that correspond to an ideal wave lattice, it would be important to learn how to describe its possible *defects*. Such defects can be associated with local patches that have *different symmetry*. In our opinion, the description of patterns with such defects may be related with a bifurcation analysis of small-amplitude *quasiperiodic* (QP) stationary solutions of model equations.

0305-4470/00/132469+19$30.00

The problem of construction and analysis of QP patterns has been actively discussed in the physics of nonlinear phenomena during the last two decades. We only mention two of many fields where such patterns have been found: crystallography [4–7] and the studies of Faraday capillary ripples [8, 9].

A substantial new impetus has been given by Arnol'd, who revealed [10, 11] an intriguing and fascinating relation between, on the one hand, QP patterns in Hamiltonian systems of a certain class and, on the other hand, purely geometric Penrose constructions (non-periodic coverings of the plane by 'Penrose tilings') [12–14].

However, neither a systematic study of complex stationary solutions which arise in simple fundamental nonlinear field models, nor a determination of conditions under which local patches with nontrivial symmetry could arise in patterns of those solutions have been fulfilled yet.

In this paper we analyse small-amplitude QP field distributions for the nonlinear scalar field described by the equation

$$u_{xx} + u_{yy} + u + \varepsilon u^2 = 0 \qquad (x, y) \in \mathcal{R}^2. \tag{1}$$

This model describes, for example, stationary flows in the fluid with the given law of vorticity [15].

The technique of obtaining QP solutions of (1) is as follows. We consider the following equation in $\mathcal{R}^4$:

$$\left(k_1\frac{\partial}{\partial\varphi_1} + k_3\frac{\partial}{\partial\varphi_3}\right)^2 u + \left(k_2\frac{\partial}{\partial\varphi_2} + k_4\frac{\partial}{\partial\varphi_4}\right)^2 u + u + \varepsilon u^2 = 0 \qquad (\varphi_1, \ldots, \varphi_4) \in \mathcal{R}^4 \tag{2}$$

where $k_1, \ldots, k_4$ are real numbers. Note that the set $\{k_j\}$ is often treated as a vector that specifies a 4D wave lattice. Equation (2) arises when we introduce a 4D space of variables $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ and treat them as dependent on the original variables x, y: $\varphi_j = \varphi_j(x, y)$; namely, we define them as follows:

$$\varphi_1 = k_1 x \qquad \varphi_2 = k_2 y \qquad \varphi_3 = k_3 x \qquad \varphi_4 = k_4 y. \tag{3}$$

Then

$$\frac{\partial}{\partial x} = \sum_j \frac{\partial\varphi_j}{\partial x}\frac{\partial}{\partial\varphi_j} \equiv k_1\frac{\partial}{\partial\varphi_1} + k_3\frac{\partial}{\partial\varphi_3}$$

and

$$\frac{\partial}{\partial y} = \sum_j \frac{\partial\varphi_j}{\partial y}\frac{\partial}{\partial\varphi_j} \equiv k_2\frac{\partial}{\partial\varphi_2} + k_4\frac{\partial}{\partial\varphi_4}.$$

Thus, if any function $u(\varphi_1, \ldots, \varphi_4)$ is a solution of equation (2), then, using relations (3), one obtains a corresponding solution of the basic equation (1). In other words, a solution of the latter can be treated as *a projection* of the 4D solution of equation (2) onto the (x, y)-plane; this projecting is performed by restricting the φ_j to $\varphi_3 = \frac{k_3}{k_1}\varphi_1$, $\varphi_4 = \frac{k_4}{k_2}\varphi_2$ and putting $x = \varphi_1/k_1$ and $y = \varphi_2/k_2$.

We will consider a class of 2D solutions of the basic equation generated by 4D strictly 2π*-periodic* in $\varphi_1, \ldots, \varphi_4$ solutions of equation (2):

$$u(\ldots, \varphi_j + 2\pi, \ldots) = u(\ldots, \varphi_j, \ldots) \qquad j = 1 \ldots 4. \tag{4}$$

Generically, 2D solutions of that class are QP functions of x and y.

Thus, the problem of finding QP solutions of equation (1) is reformulated as a problem of finding 2π-periodic solutions of the 'extended' 4D equation (2) followed by projecting these solutions back onto the (x, y)-plane by means of relations (3).

Of course, this way of finding QP solutions is not unique. But it has a substantial attractive feature: it directly follows the same logic as used by Arnol'd and other authors, who obtain QP tilings of the plane as projections of cells of a multidimensional periodic lattice that are intersected by an 'irrationally oriented' section plane, onto this plane. As was shown by Arnol'd (see, for instance, [10]), the analysis of dispersion relations in the space of wave numbers k_j may establish reasons *why* local patches with nontrivial symmetry arise.

There exists another reason to develop the above scheme of constructing QP solutions: such solutions can be expanded along a QP basis, which can also be built following the same procedure. Actually, for the Laplace operator in $\mathcal{R}^2$ one can consider the eigenvalue problem in the class of QP functions that are projections of periodic functions of phase variables [16]. The number of phase variables to be introduced is not prescribed. For example, in [16] we study bases of functions built on a 2D wave lattice (k_1, k_2): $\varphi_1 = k_1 x$, $\varphi_2 = k_2 y$. In this case, the quasiperiodicity of the basis functions (modes) is related to an infinite (but countable) degeneration of eigenvalues of the Laplace operator w.r.t. the parameters of the wave lattice. Sometimes, one says that such QP modes are defined on a foliation of a family of tori in the phase space (as linear envelopes of corresponding periodic solutions).

As to 2π-periodic solutions of the 4D equation (2), they are constructed in the form of asymptotic expansions for $\varepsilon \ll 1$. To that purpose we will use a generalization of the approach presented in [17]; it can be viewed as a development of the multi-scale expansion methods (exposed, for instance, in [18]).

The paper is organized as follows. In section 2 we define asymptotic expansions of QP solutions and consider the zero order. First order is analysed in section 3; one- and two-dimensional spatial resonances are discussed; the example of 2D resonance QP pattern is studied. QP solutions in the case of 1D resonance are analysed in section 4. In conclusion we discuss a possible generalization of our approach to QP patterns in nonlinear field models. An appendix presents a simple analysis of nontrivial symmetries in the 2D resonance QP pattern obtained in section 3.

2. Asymptotical expansions: zero order

Let us look for solutions of (2) and components k_j of the wave lattice vector in the form of power series in ε:

$$u(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \sum_{n=0} \varepsilon^n u_n(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \tag{5}$$

$$k_j = \sum_{n=0} \varepsilon^n k_j^{(n)} \qquad j = 1, 2, 3, 4. \tag{6}$$

Substituting these expressions into equation (2) generates a chain of linear inhomogeneous equations:

$$\hat{\boldsymbol{L}}_0 u_0 \equiv ([D_{13}^{(0)}]^2 + [D_{24}^{(0)}]^2 + 1)u_0 = 0 \tag{7}$$

$$\hat{\boldsymbol{L}}_0 u_1 = -u_0^2 + 2(D_{13}^{(0)} D_{13}^{(1)} + D_{24}^{(0)} D_{24}^{(1)})u_0 \tag{8}$$

$$\hat{\boldsymbol{L}}_0 u_2 = -2u_0 u_1 - 2(D_{13}^{(0)} D_{13}^{(2)} + D_{24}^{(0)} D_{24}^{(2)})u_0 - ([D_{13}^{(1)}]^2 + [D_{24}^{(1)}]^2)u_0 \\ -2(D_{13}^{(0)} D_{13}^{(1)} + D_{24}^{(0)} D_{24}^{(1)})u_1 \tag{9}$$

and so on. Here we denoted

$$D_{13}^{(n)} \equiv k_1^{(n)} \frac{\partial}{\partial \varphi_1} + k_3^{(n)} \frac{\partial}{\partial \varphi_3} \qquad D_{24}^{(n)} \equiv k_2^{(n)} \frac{\partial}{\partial \varphi_2} + k_4^{(n)} \frac{\partial}{\partial \varphi_4}. \tag{10}$$

Now we dwell on the equation of zero order (7). Studying this equation, for the sake of brevity, let us write k_j instead of $k_j^{(0)}$.

Let us consider a class of solutions of (7) that (a) are 2π*-periodic* in φ_j $(j = 1 \ldots 4)$ and (b) are projected onto *even* both in x and y 2D functions (by means of formulae (3)).

In that class, it is natural to seek solutions of (7) in the form

$$u_0(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = A\cos\varphi_1\cos\varphi_2\cos\varphi_3\cos\varphi_4 + B\sin\varphi_1\cos\varphi_2\sin\varphi_3\cos\varphi_4 \\ +C\cos\varphi_1\sin\varphi_2\cos\varphi_3\sin\varphi_4 + D\sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4 \qquad (11)$$

(it is easy to see that each term in this formula is projected onto an even function in (x, y)). Substituting this expression into equation (7), we find that the coefficients A, B, C, D must simultaneously obey the following four relations:

$$\mathcal{K}^{(+1,+1)}\mathcal{A}^{(+1,+1)} = 0 \qquad \mathcal{K}^{(-1,-1)}\mathcal{A}^{(-1,-1)} = 0 \\ \mathcal{K}^{(+1,-1)}\mathcal{A}^{(+1,-1)} = 0 \qquad \mathcal{K}^{(-1,+1)}\mathcal{A}^{(-1,+1)} = 0 \qquad (12)$$

where

$$\mathcal{K}^{(\sigma_1,\sigma_2)} \equiv -\sum_{j=1}^{4} k_j^2 + 2\sigma_1 k_1 k_3 + 2\sigma_2 k_2 k_4 + 1 \qquad \sigma_{1,2} = \pm 1 \qquad (13)$$

$$\mathcal{A}^{(\sigma_1,\sigma_2)} \equiv A + \sigma_1 B + \sigma_2 C + \sigma_1\sigma_2 D. \qquad (14)$$

In the most general case, only one $\mathcal{K}^{(\sigma_1,\sigma_2)}$ (for a certain choice of $\sigma_1 = \bar{\sigma}_1$ and $\sigma_2 = \bar{\sigma}_2$) vanishes; $\mathcal{K}^{(\bar{\sigma}_1,\bar{\sigma}_2)} = 0$. Since this is a relation between the k_j, this provides us with a dispersion relation. Then the corresponding expression $\mathcal{A}^{(\bar{\sigma}_1,\bar{\sigma}_2)}$ does not vanish, while the three other $\mathcal{A}^{(\sigma_1,\sigma_2)}$ for different σ_1 and σ_2 must simultaneously vanish. Analysing the four possible cases, we see that always $|A| = |B| = |C| = |D|$; let all these amplitudes be equal to 1.

After obvious trigonometry, we find that in all the four cases solution (11) and the corresponding dispersion relation can be written in very simple form:

$$u_0^{(\sigma_1,\sigma_2)}(\varphi_1, \ldots, \varphi_4) = \cos(\varphi_1 - \sigma_1\varphi_3)\cos(\varphi_2 - \sigma_2\varphi_4) \qquad (15)$$

$$\mathcal{F}^{(\sigma_1,\sigma_2)}: \quad (k_1 - \sigma_1 k_3)^2 + (k_2 - \sigma_2 k_4)^2 = 1. \qquad (16)$$

Using relations (3), we see that these solutions (2π-periodic in the 4D phase space) are projected onto the solutions

$$u_0^{(\sigma_1,\sigma_2)}(x, y) = \cos(k_1 - \sigma_1 k_3)x\ \cos(k_2 - \sigma_2 k_4)y \qquad (17)$$

which are also strictly periodic, but now in x and y (with periods $2\pi/(k_1 - \sigma_1 k_3)$ and $2\pi/(k_2 - \sigma_2 k_4)$, respectively).

Note now that the dispersion surfaces (16) for different pairs (σ_1, σ_2) can *intersect* along dispersion submanifolds of lower dimension. On such submanifolds different solutions of type (15) coexist. Returning to conditions (12), we see that in such cases the coefficients A, B, C, D must obey only two equations of the form $\mathcal{A}^{(\sigma_1,\sigma_2)} = 0$; as a result, the general expression (11) now includes a continuous parameter. As one might expect, such a solution can be rewritten as a linear combination of two corresponding elementary solutions (15) taken with arbitrary relative weight.

But, unlike the elementary solutions, which are projected onto periodic functions (17), this combined function is (generically) projected onto a *quasi-periodic* function in (x, y).

Thus, in zero order we have obtained a family of QP solutions of the basic equation (1). They exist on a corresponding dispersion submanifold, where both elementary components have the same wave numbers $\{k_j\}$.

In particular, the solution

$$u_0(\varphi_1, \ldots, \varphi_4) = \cos(\varphi_1 + \varphi_3)\cos(\varphi_2 + \varphi_4) + \gamma_1\cos(\varphi_1 - \varphi_3)\cos(\varphi_2 - \varphi_4) \qquad (18)$$

which generates the 2D QP solution

$$u_0(x,y) = \cos(k_1+k_3)x\cos(k_2+k_4)y + \gamma_1\cos(k_1-k_3)x\cos(k_2-k_4)y \tag{19}$$

exists on the following dispersion surface:

$$\mathcal{F}^{(+1,+1)} \cap \mathcal{F}^{(-1,-1)}: \quad \sum_{j=1}^{4} k_j^2 = 1 \qquad k_1k_3 + k_2k_4 = 0 \tag{20}$$

and the solution

$$u_0(\varphi_1,\ldots,\varphi_4) = \cos(\varphi_1-\varphi_3)\cos(\varphi_2+\varphi_4) + \gamma_2\cos(\varphi_1+\varphi_3)\cos(\varphi_2-\varphi_4) \tag{21}$$

which generates the 2D QP solution

$$u_0(x,y) = \cos(k_1-k_3)x\cos(k_2+k_4)y + \gamma_1\cos(k_1+k_3)x\cos(k_2-k_4)y \tag{22}$$

exists on the dispersion surface

$$\mathcal{F}^{(+1,-1)} \cap \mathcal{F}^{(-1,+1)}: \quad \sum_{j=1}^{4} k_j^2 = 1 \qquad k_1k_3 - k_2k_4 = 0. \tag{23}$$

Remark. More strictly, the solutions (19) and (22) are QP almost for all values of k_j except for a countable set

$$k_1^2 = \frac{-r_2}{(r_1-r_2)(1-r_1r_2)} \qquad k_2^2 = \frac{r_1}{(r_1-r_2)(1-r_1r_2)} \qquad k_3 = r_1k_1 \qquad k_4 = r_2k_2$$
$$r_1, r_2 \text{ are rational} \tag{24}$$

for the first type of solutions (19), and

$$k_1^2 = \frac{r_2}{(r_1+r_2)(1+r_1r_2)} \qquad k_2^2 = \frac{r_1}{(r_1+r_2)(1+r_1r_2)} \qquad k_3 = r_1k_1 \qquad k_4 = r_2k_2$$
$$r_1, r_2 \text{ are rational} \tag{25}$$

for the second type (22). For such k_j the solutions (19) and (22) become *strictly* periodic in (x, y): since in these solutions the periods of the two terms in each space direction ($T_x^{(1)} = 2\pi/(k_1+k_3)$ and $T_x^{(2)} = 2\pi/(k_1-k_3)$; $T_y^{(1)} = 2\pi/(k_2+k_4)$ and $T_y^{(2)} = 2\pi/(k_2-k_4)$) are commensurable, there exist common periods for both the terms.

Note that if the numbers r_1, r_2 are treated as arbitrary numbers then relations (24), (25) simply give us a parametric expression for the 2D dispersion manifolds (20), (23).

Let us return to the intersection of the dispersion surfaces $\mathcal{F}^{(+1,+1)}$ and $\mathcal{F}^{(-1,-1)}$. Due to the form of the relation (16), it is natural to introduce angle variables to specify the vector of the wave lattice on this dispersion subspace. In the case of $\sigma_1 = \sigma_2$ they can be defined as follows:

$$k_1 + k_3 = \cos\phi \qquad k_1 - k_3 = \cos\psi$$
$$k_2 + k_4 = \sin\phi \qquad k_2 - k_4 = \sin\psi.$$

Then

$$k_1 = \cos\left(\frac{\phi-\psi}{2}\right)\cos\left(\frac{\phi+\psi}{2}\right) \qquad k_2 = \cos\left(\frac{\phi-\psi}{2}\right)\sin\left(\frac{\phi+\psi}{2}\right) \tag{26}$$

$$k_3 = -\sin\left(\frac{\phi-\psi}{2}\right)\sin\left(\frac{\phi+\psi}{2}\right) \qquad k_4 = \sin\left(\frac{\phi-\psi}{2}\right)\cos\left(\frac{\phi+\psi}{2}\right). \tag{27}$$

The numbers k_j thus defined always (for all ϕ and ψ) satisfy the dispersion relation (20) that arises at the intersection of $\mathcal{F}^{(+1,+1)}$ and $\mathcal{F}^{(-1,-1)}$. Then the corresponding 2D QP solution can be written in the simple form:

$$u_0(x, y) = \cos([\cos\phi]x)\ \cos([\sin\phi]y) + \gamma\cos([\cos\psi]x)\ \cos([\sin\psi]y). \tag{28}$$

In the case of intersection of the surfaces $\mathcal{F}^{(+1,-1)}$ and $\mathcal{F}^{(-1,+1)}$ the angle variables are introduced as follows:

$$k_1 + k_3 = \cos\phi \qquad k_1 - k_3 = \cos\psi$$
$$k_2 - k_4 = \sin\phi \qquad k_2 + k_4 = \sin\psi.$$

Then k_j are also defined by expressions (26), (27) except for the opposite sign of the right-hand side of the formula for k_4. The numbers k_j thus defined satisfy the dispersion relation (23) for all ϕ and ψ. Note that the corresponding QP solution is the same as in the first case: it is also expressed by equation (28).

In what follows we restrict ourselves to the *second* simplest mode defined by expressions (21), (23) and let $\gamma_2 = 1$. Thus, as a solution of the zero-order equation (7) we take a symmetric combination of elementary periodic solutions $u_0^{(+1,-1)} + u_0^{(-1,+1)}$.

Finally, let us introduce an additional notation. As is seen from expressions (26), (27),

$$k_1 = \boldsymbol{K}_{12}\cos\alpha \qquad k_2 = \boldsymbol{K}_{12}\sin\alpha \qquad k_3 = \boldsymbol{K}_{34}\cos\beta \qquad k_4 = \boldsymbol{K}_{34}\sin\beta \tag{29}$$

where

$$\boldsymbol{K}_{12} \equiv \cos\left(\frac{\phi-\psi}{2}\right) \qquad \boldsymbol{K}_{34} \equiv -\sin\left(\frac{\phi-\psi}{2}\right) \tag{30}$$

$$\alpha \equiv \frac{\phi+\psi}{2} \qquad \beta \equiv \frac{\pi}{2} - \frac{\phi+\psi}{2}. \tag{31}$$

So, we can treat the numbers k_1, k_2 as components of the 2D wave ‘subvector’ that is a projection of the full 4D vector of the wave lattice onto the plane (k_1, k_2); analogously, the numbers k_3, k_4 are components of the second 2D ‘subvector’ that is a projection of the full 4D vector onto the plane (k_3, k_4). Note that

$$\boldsymbol{K}_{12}^2 + \boldsymbol{K}_{34}^2 = 1 \qquad \beta + \alpha = \pi/2. \tag{32}$$

3. Asymptotical expansions: first order. 2D spatial resonance

According to equation (8), in the next order in $\varepsilon \ll 1$ the main mode (21), (23) generates the following expression for the function $u_1(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$:

$$\begin{aligned} u_1 = -\frac{1}{2} - \frac{1}{4}&\left\{\frac{\cos 2(\varphi_1-\varphi_3)}{1-4(k_1-k_3)^2} + \frac{\cos 2(\varphi_1+\varphi_3)}{1-4(k_1+k_3)^2}\right\} \\ -\frac{1}{4}&\left\{\frac{\cos 2(\varphi_2+\varphi_4)}{1-4(k_2+k_4)^2} + \frac{\cos 2(\varphi_2-\varphi_4)}{1-4(k_2-k_4)^2}\right\} \\ -\frac{1}{4}&\left\{\frac{\cos 2(\varphi_1-\varphi_3)\cos 2(\varphi_2+\varphi_4)}{1-4(k_1-k_3)^2-4(k_2+k_4)^2} + \frac{\cos 2(\varphi_1+\varphi_3)\cos 2(\varphi_2-\varphi_4)}{1-4(k_1+k_3)^2-4(k_2-k_4)^2}\right\} \\ -\frac{1}{2}&\left\{\frac{\cos 2\varphi_1\cos 2\varphi_2}{1-4(k_1^2+k_2^2)} + \frac{\cos 2\varphi_1\cos 2\varphi_4}{1-4(k_1^2+k_4^2)} + \frac{\cos 2\varphi_3\cos 2\varphi_2}{1-4(k_3^2+k_2^2)} + \frac{\cos 2\varphi_3\cos 2\varphi_4}{1-4(k_3^2+k_4^2)}\right\} \end{aligned} \tag{33}$$

and two relations:

$$\begin{aligned} &(k_1-k_3)(k_1^{(1)}-k_3^{(1)}) + (k_2+k_4)(k_2^{(1)}+k_4^{(1)}) = 0 \\ &(k_1+k_3)(k_1^{(1)}+k_3^{(1)}) + (k_2-k_4)(k_2^{(1)}-k_4^{(1)}) = 0 \end{aligned} \tag{34}$$

which arise as a result of eliminating secular terms in the right-hand side of equation (8). In the space $\{(k_1, k_2, k_3, k_4)\}$ these relations define a 2D manifold of admissible variations of the 4D vector (k_1, k_2, k_3, k_4), which belongs to the 2D manifold (23). This becomes evident if we rewrite relations (34) in the following form:

$$k_1 k_1^{(1)} + k_2 k_2^{(1)} + k_3 k_3^{(1)} + k_4 k_4^{(1)} = 0 \qquad k_3 k_1^{(1)} + k_1 k_3^{(1)} = k_4 k_2^{(1)} + k_2 k_4^{(1)}. \tag{35}$$

Expression (33) for the function u_1 provides the following two types of small divisors:

$$(I): \quad (k_1 \mp k_3)^2 = \tfrac{1}{4} \qquad (k_2 \pm k_4)^2 = \tfrac{1}{4} \tag{36}$$

$$(II): \quad k_1^2 + k_2^2 = \tfrac{1}{4} \qquad k_1^2 + k_4^2 = \tfrac{1}{4} \qquad k_3^2 + k_2^2 = \tfrac{1}{4} \qquad k_3^2 + k_4^2 = \tfrac{1}{4}. \tag{37}$$

The first type of small divisor may be related with inner 1D spatial resonances (since the projection of the resonance term is a function of one space variable only: either x or y), whereas the second type may be related with inner 2D spatial resonance (since the projection of the resonance term is a function of both the space variables) of waves in nonlinear media.

Let us first consider the 2D resonance related with the small divisor $k_1^2 + k_2^2 = \frac{1}{4}$. It may occur if the *three* dispersion manifolds intersect:

$$K^2 \equiv \sum_{j=1}^{4} k_j^2 = 1 \qquad k_3 k_1 = k_2 k_4 \qquad k_1^2 + k_2^2 = \tfrac{1}{4}. \tag{38}$$

This intersection takes place on a 1D resonance manifold in the 4D space $\{(k_1, k_2, k_3, k_4)\}$.

To eliminate that small divisor, let us redefine the zero approximation:

$$\tilde{u}_0 = u_0 + \rho \cos 2\varphi_1 \cos 2\varphi_2. \tag{39}$$

Here ρ is a parameter (magnitude of the resonance mode), which will be defined below. The change (39) leads to the following equation for the first approximation:

$$\begin{aligned}\hat{\boldsymbol{L}}_0 \tilde{u}_1 = &-u_0^2 - 2\rho u_0 \cos 2\varphi_1 \cos 2\varphi_2 - \tfrac{1}{4}\rho^2 (1 + \cos 4\varphi_1 + \cos 4\varphi_2 + \cos 4\varphi_1 \cos 4\varphi_2) \\ &+ 2(D_{13}^{(0)} D_{13}^{(1)} + D_{24}^{(0)} D_{24}^{(1)})(u_0 + \rho \cos 2\varphi_1 \cos 2\varphi_2).\end{aligned} \tag{40}$$

In the right-hand side of this equation secular terms (proportional to u_0 and $\cos 2\varphi_1 \cos 2\varphi_2$) can be produced by the terms u_0^2 and $\rho u_0 \cos 2\varphi_1 \cos 2\varphi_2$:

$$u_0^2 \Rightarrow \tfrac{1}{2} \cos 2\varphi_1 \cos 2\varphi_2 + \cdots \qquad \rho u_0 \cos 2\varphi_1 \cos 2\varphi_2 \Rightarrow \tfrac{1}{4} \rho u_0 + \cdots. \tag{41}$$

Elimination of these terms gives us three relations, which include the magnitude of the resonance mode ρ:

$$\begin{aligned}&(k_1 - k_3)(k_1^{(1)} - k_3^{(1)}) + (k_2 + k_4)(k_2^{(1)} + k_4^{(1)}) = \tfrac{1}{4}\rho \\ &(k_1 + k_3)(k_1^{(1)} + k_3^{(1)}) + (k_2 - k_4)(k_2^{(1)} - k_4^{(1)}) = \tfrac{1}{4}\rho \\ &k_1 k_1^{(1)} + k_2 k_2^{(1)} = \frac{1}{16\rho}.\end{aligned} \tag{42}$$

Then the solution of equation (40) takes the form

$$\begin{aligned}\tilde{u}_1 = u_1' - \frac{1}{2}\rho^2 - \frac{1}{2}\rho^2 &\left\{ \frac{\cos 4\varphi_1}{1 - (4k_1)^2} + \frac{\cos 4\varphi_2}{1 - (4k_2)^2} + \frac{\cos 4\varphi_1 \cos 4\varphi_2}{1 - (4k_1)^2 - (4k_2)^2} \right\} \\ -\frac{1}{2} &\left\{ \frac{\cos(\varphi_1 - \varphi_3)\cos(3\varphi_2 - \varphi_4)}{1 - (k_1 - k_3)^2 - (3k_2 - k_4)^2} + \frac{\cos(\varphi_1 + \varphi_3)\cos(3\varphi_2 + \varphi_4)}{1 - (k_1 + k_3)^2 - (3k_2 + k_4)^2} \right\} \\ -\frac{1}{2} &\left\{ \frac{\cos(3\varphi_1 - \varphi_3)\cos(\varphi_2 - \varphi_4)}{1 - (3k_1 - k_3)^2 - (k_2 - k_4)^2} + \frac{\cos(3\varphi_1 + \varphi_3)\cos(\varphi_2 + \varphi_4)}{1 - (3k_1 + k_3)^2 - (k_2 + k_4)^2} \right\} \\ -\frac{1}{2} &\left\{ \frac{\cos(3\varphi_1 - \varphi_3)\cos(3\varphi_2 - \varphi_4)}{1 - (3k_1 - k_3)^2 - (3k_2 - k_4)^2} + \frac{\cos(3\varphi_1 + \varphi_3)\cos(3\varphi_2 + \varphi_4)}{1 - (3k_1 + k_3)^2 - (3k_2 + k_4)^2} \right\}.\end{aligned} \tag{43}$$

Here u_1' is determined by formula (33) where the term with the eliminated small divisor should be omitted.

Using the notation introduced at the end of the previous section, we find that

$$k_1 = \tfrac{1}{2}\cos\alpha \qquad k_2 = \tfrac{1}{2}\sin\alpha \qquad k_3 = \frac{\sqrt{3}}{2}\sin\alpha \qquad k_4 = \frac{\sqrt{3}}{2}\cos\alpha. \tag{44}$$

on the resonance manifold (38). Then the 2D solution obtained as a projection of the renormalized 4D resonance mode (39) can be written as follows:

$$\begin{aligned} u_0^{\mathrm{res}}(x, y; \alpha) = {} & \cos\left[\cos\left(\alpha + \frac{\pi}{3}\right)\right] x \cos\left[\sin\left(\alpha + \frac{\pi}{3}\right)\right] y \\ & + \cos\left[\cos\left(\alpha - \frac{\pi}{3}\right)\right] x \ \cos\left[\sin\left(\alpha - \frac{\pi}{3}\right)\right] y + \rho\cos[\cos\alpha]\, x\cos[\sin\alpha]\, y. \end{aligned} \tag{45}$$

It is QP w.r.t. both x and y up to a countable set of exactly periodic 2D solutions that are specified by the condition

$$\sqrt{3}\tan\alpha = r \qquad (r \text{ is a rational number}). \tag{46}$$

Note also that this solution has a parameter (α). In expressions (29), suppose that

$$\boldsymbol{K}_{12} = \boldsymbol{K}_{12}^0 + \varepsilon\boldsymbol{K}_{12}^{(1)} \qquad \boldsymbol{K}_{34} = \boldsymbol{K}_{34}^0 + \varepsilon\boldsymbol{K}_{34}^{(1)} \tag{47}$$

$$\alpha = \alpha^0 + \varepsilon\alpha^{(1)} \qquad \beta = \beta^0 + \varepsilon\beta^{(1)} \tag{48}$$

then we can write relations (42) in the form

$$\begin{aligned} & k_1k_1^{(1)} + k_2k_2^{(1)} = \tfrac{1}{2}\boldsymbol{K}_{12}^{(1)} = \frac{1}{16\rho} \\ & k_3k_3^{(1)} + k_4k_4^{(1)} = \frac{\sqrt{3}}{2}\boldsymbol{K}_{34}^{(1)} = \frac{1}{2}\left(\rho - \frac{1}{8\rho}\right) \\ & \beta^{(1)} = \alpha^{(1)}. \end{aligned} \tag{49}$$

These expressions explicitly show that the variations of the modules of the 2D vectors (k_1, k_2) and (k_3, k_4) are specified by the parameter ρ (the magnitude of the resonance mode), whereas the difference $\beta - \alpha$ does not depend on it: in the first order in $\varepsilon \ll 1$

$$\beta - \alpha = \beta^0 - \alpha^0 = \frac{\pi}{2} - 2\alpha^0. \tag{50}$$

Considering the location of small divisors in expressions (33) and (43) with respect to the resonance manifold (38), it is easy to prove that their coincidence occurs for the following values of α:

$$\sqrt{3}\tan\alpha = 0, \pm\tfrac{1}{3}, \pm 1, \pm 3, \pm\infty \tag{51}$$

which are associated with solutions that either are strictly periodic w.r.t. x and y, or represent a degenerate case ($k_2 = k_3 = 0$, $k_1 \neq 0$, $k_4 \neq 0$ or $k_1 = k_4 = 0$, $k_2 \neq 0$, $k_3 \neq 0$).

Note also that the 2D resonances with $j = 1$, $j' = 2$ and $j = 3$, $j' = 4$ are equivalent in the sense of pattern topology.

Now we describe how the QP 2D resonance solution $u_0^{\mathrm{res}}(x, y; \alpha)$ (45) transforms as the parameter α varies. Let $\rho = 1$. At $\alpha = \pi/6$ the solution u_0^{res} has a simple pattern: a regular lattice with hexagonal symmetry.

In the 'symmetric case' (at $\alpha = \pi/4$) this function can be written in the form

$$\begin{aligned} & u(\xi, \eta) = \cos(\xi)\cos(\eta) + \cos(S_1\xi)\cos(S_2\eta) + \cos(S_2\xi)\cos(S_1\eta) \\ & S_1 = \frac{\sqrt{3}+1}{2} \qquad S_2 = \frac{\sqrt{3}-1}{2} \qquad \xi \equiv x\sqrt{2} \qquad \eta \equiv y\sqrt{2}. \end{aligned} \tag{52}$$

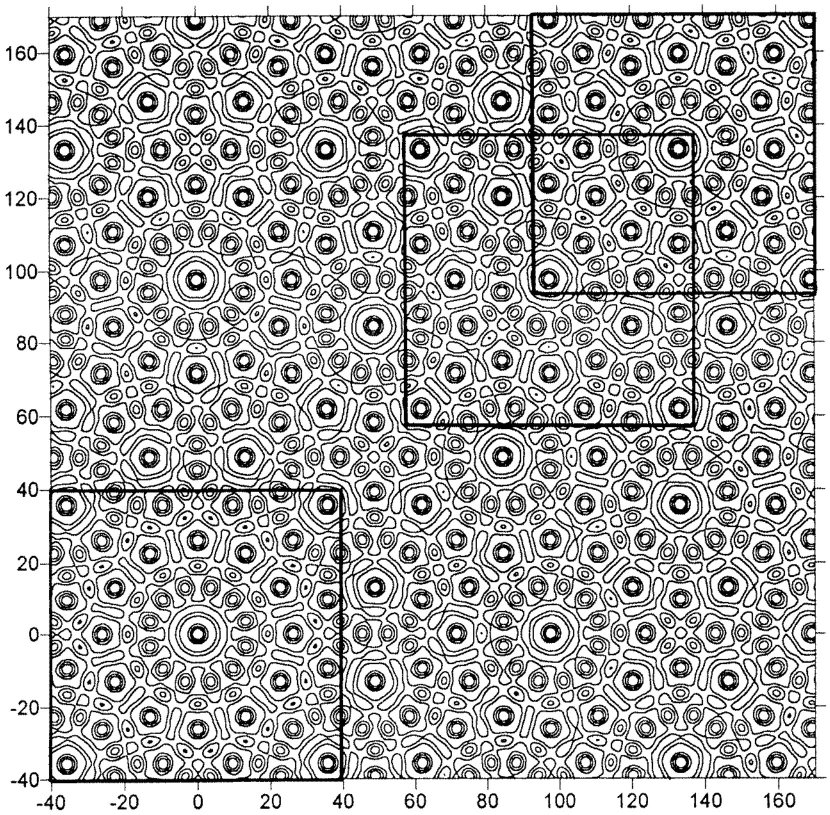


Figure 1. The QP 2D resonance solution ($\alpha = \pi/4$). Three squares selected in the pattern are drawn in detail in figures 2(*a*)–(*c*).

Its pattern is characterized by the global symmetry axis of 12th order w.r.t. the origin (figure 1). Since the function is QP, this pattern must approximately reproduce itself w.r.t. infinitely many points of the plane; this is illustrated in figure 2.

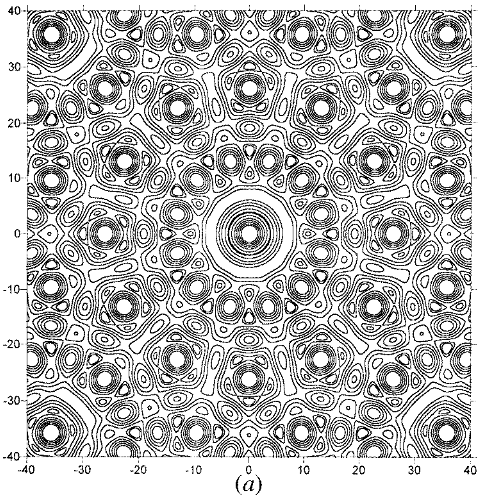
At the same time, in the relief of u_0^{res} there are distinct local patches with approximate order-5 symmetry. To recognize such fragments clearly, we introduced the following gradient dynamical system:

$$\dot{x} = -\frac{\partial u_0^{\mathrm{res}}}{\partial x} \qquad \dot{y} = -\frac{\partial u_0^{\mathrm{res}}}{\partial y} \tag{53}$$

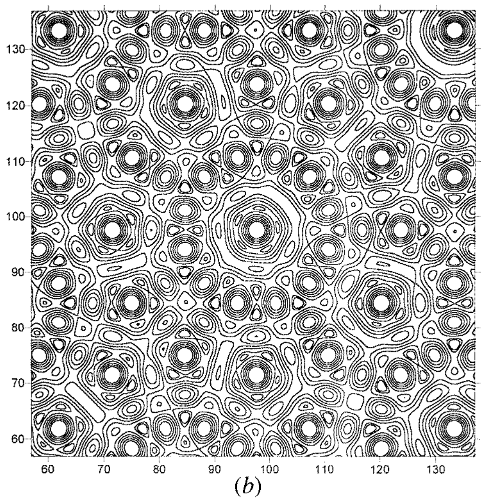
and numerically found stationary points and lines of transition from local minima and maxima to saddle-type points. This decoration of the pattern proved both a nontraditional symmetry of its fragments and revealed analogues of the Ammann lattice [6, 11].

Moreover, it occurred that around the points where the initial pattern with order-12 symmetry is almost reproduced there always exist areas inside which the order-5 symmetry evidently prevails. A simple analysis of the interference between the symmetries of 12th and 5th order in the function (52) is presented in the appendix.

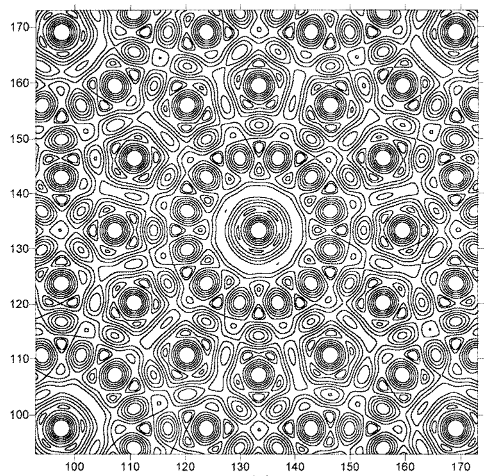
Finally, let us describe the behaviour of the pattern if $\pi/6 < \alpha < \pi/4$. As α begins moving from $\pi/6$ (the simple hexagonal lattice), on the background of this pattern there arises another lattice; its cells have the same symmetry but much larger size. With further growth of α this second lattice shrinks; the boundaries of its cells deform and gain their own inner



(*a*)



(*b*)



(*c*)

Figure 2. QP almost-replication of the pattern. (*a*) The pattern with exact 12-fold symmetry around the origin. (*b*), (*c*) Two domains of the plane (see the two right-top squares in figure 1) where this pattern is approximately reproduced with different exactitude.

structure. This process is illustrated in figure 3. As α varies from $\pi/4$ to $\pi/3$ the inverse scenario takes place.

4. 1D spatial resonance

Consider another case: a 1D spatial resonance, which is generated by the small divisor $(k_1^0 - k_3^0)^2 = \frac{1}{4}$ in (33) (a 2D projection of the resonance term $\cos 2(\varphi_1 - \varphi_3)$ is actually a function of x only). The associated 1D resonance manifold in the 4D space $\{(k_1, k_2, k_3, k_4)\}$ is specified by the intersection of the three dispersion manifolds

$$k_1^2 + k_2^2 + k_3^2 + k_4^2 = 1 \qquad k_3 k_1 = k_2 k_4 \qquad (k_1 - k_3)^2 = \tfrac{1}{4}. \tag{54}$$

As before, to eliminate that small divisor, we redefine the zero approximation:

$$\tilde{u}_0 = u_0 + \rho \cos 2(\varphi_1 - \varphi_3). \tag{55}$$

This substitution leads to the following equation for the first approximation:

$$\hat{\boldsymbol{L}}_0 \tilde{u}_1 = -u_0^2 - 2\rho u_0 \cos 2(\varphi_1 - \varphi_3) + \tfrac{1}{2}\rho^2 (1 + \cos 4(\varphi_1 - \varphi_3))$$

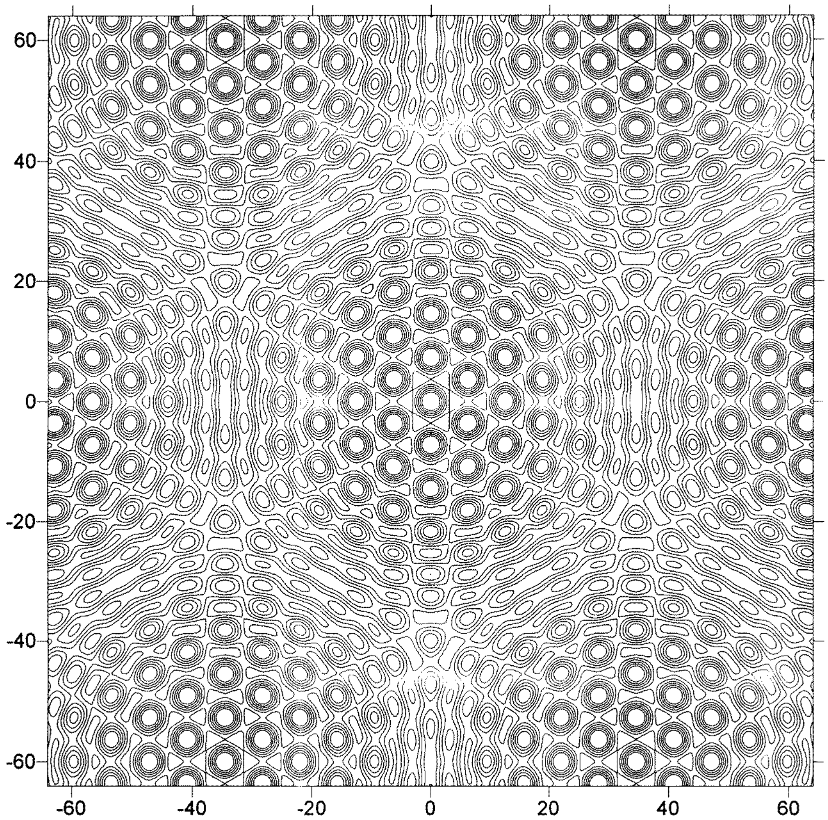


Figure 3. Level lines for the pattern in the case of 2D resonance on the stage of two overlapping hexagonal patterns with different spatial size ($\alpha = 63^\circ$).

$$-2(D_{13}^{(0)} D_{13}^{(1)} + D_{24}^{(0)} D_{24}^{(1)})(u_0 + \rho \cos 2(\varphi_1 - \varphi_3)). \tag{56}$$

In the right-hand side of this equation the terms u_0^2 and $\rho u_0 \cos 2(\varphi_1 - \varphi_3)$ are secular (proportional to u_0 and $\cos 2(\varphi_1 - \varphi_3)$):

$$u_0^2 \Rightarrow \tfrac{1}{4} \cos 2(\varphi_1 - \varphi_3) + \cdots \qquad u_0 \cos 2(\varphi_1 - \varphi_3) \Rightarrow \tfrac{1}{2} \cos(\varphi_1 - \varphi_3) \cos(\varphi_2 + \varphi_4) + \cdots . \tag{57}$$

Elimination of these terms also gives us three relations:

$$\begin{aligned} &(k_1 - k_3)(k_1^{(1)} - k_3^{(1)}) + (k_2 + k_4)(k_2^{(1)} + k_4^{(1)}) = \tfrac{1}{2}\rho \\ &(k_1 + k_3)(k_1^{(1)} + k_3^{(1)}) + (k_2 - k_4)(k_2^{(1)} - k_4^{(1)}) = 0 \\ &(k_1 - k_3)(k^{(1)} - k_3^{(1)}) = \frac{1}{32\rho}. \end{aligned} \tag{58}$$

Equation (56) has the following solution:

$$\begin{aligned} \tilde{u}_1 = u_1' - \frac{1}{4}\rho^2 - \frac{1}{4}\rho^2 &\left\{ \frac{\cos 4\varphi_1}{1 - (4k_1)^2} + \frac{\cos 4\varphi_2}{1 - (4k_2)^2} + \frac{\cos 4\varphi_1 \cos 4\varphi_2}{1 - (4k_1)^2 - (4k_2)^2} \right\} \\ &- \frac{1}{2} \left\{ \frac{\cos(\varphi_1 - 3\varphi_3)\cos(\varphi_2 - \varphi_4)}{1 - (k_1 - 3k_3)^2 - (k_2 - k_4)^2} + \frac{\cos(3\varphi_1 - \varphi_3)\cos(\varphi_2 - \varphi_4)}{1 - (3k_1 - k_3)^2 - (k_2 - k_4)^2} \right. \\ &\left. + \frac{\cos 3(\varphi_1 - \varphi_3)\cos(\varphi_2 - \varphi_4)}{1 - 9(k_1 - k_3)^2 - (k_2 - k_4)^2} \right\}. \end{aligned} \tag{59}$$

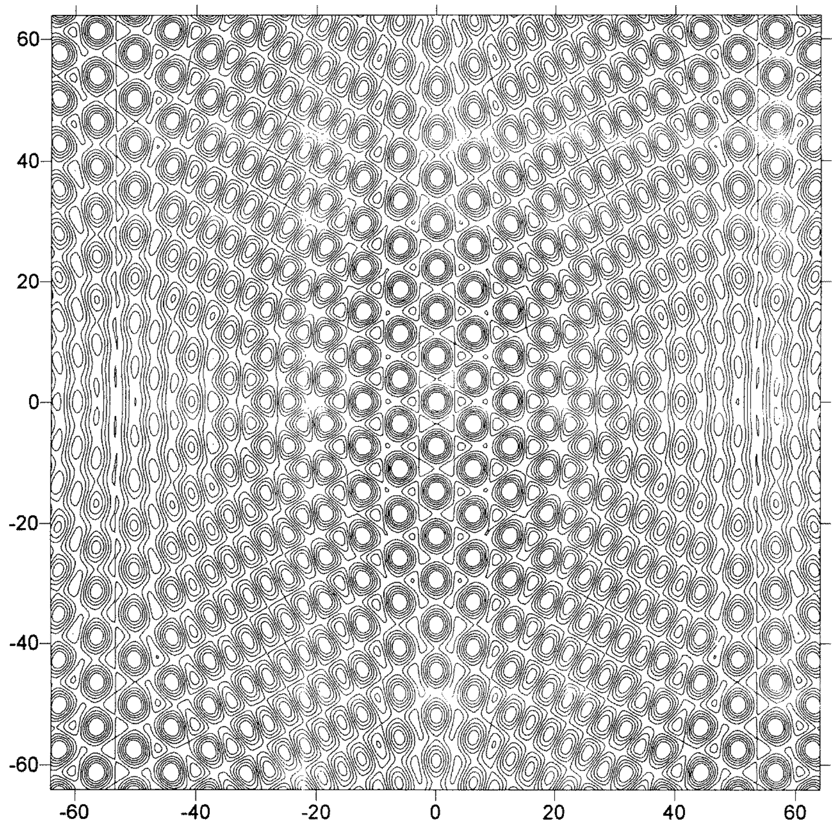


Figure 4. Level lines for the pattern in the case of 1D resonance: alternating pieces with hexagonal and quadrangular packs ($\alpha = 2°$) and stripes.

The 1D resonance manifold for this case is specified by the conditions

$$\boldsymbol{K}_{12} = \cos\gamma \qquad \boldsymbol{K}_{34} = \sin\gamma \qquad \beta = \frac{\pi}{2} - \alpha \qquad \tan(\gamma + \alpha) = \pm\sqrt{3}. \tag{60}$$

The 2D projection of the renormalized main mode (55) is QP up to a countable set of exactly periodic solutions selected by the conditions

$$\tan^2\gamma = r_1 r_2 \qquad \tan^2\alpha = r_1/r_2 \tag{61}$$

if r_1 and r_2 are related as follows:

$$\frac{r_1}{r_2}\left(\frac{1 + r_2}{1 - r_1}\right)^2 = 3. \tag{62}$$

The cases $j = 1, j' = 3$ and $j = 2, j' = 4$ are also equivalent in the sense of pattern topology.

Numerically, in the case of 1D resonance the pattern also has quite a complicated structure. Nevertheless, in the limit cases $\alpha \to 0$ and $\alpha \to \pi/6$ it also degenerates into a simple hexagonal lattice. As α is small, the pattern can be described as alternating macropatches that contain close packs of elementary cells with either hexagonal or quadrangular symmetry, respectively (figure 4). As α grows further, the pattern becomes more complicated (figure 5).

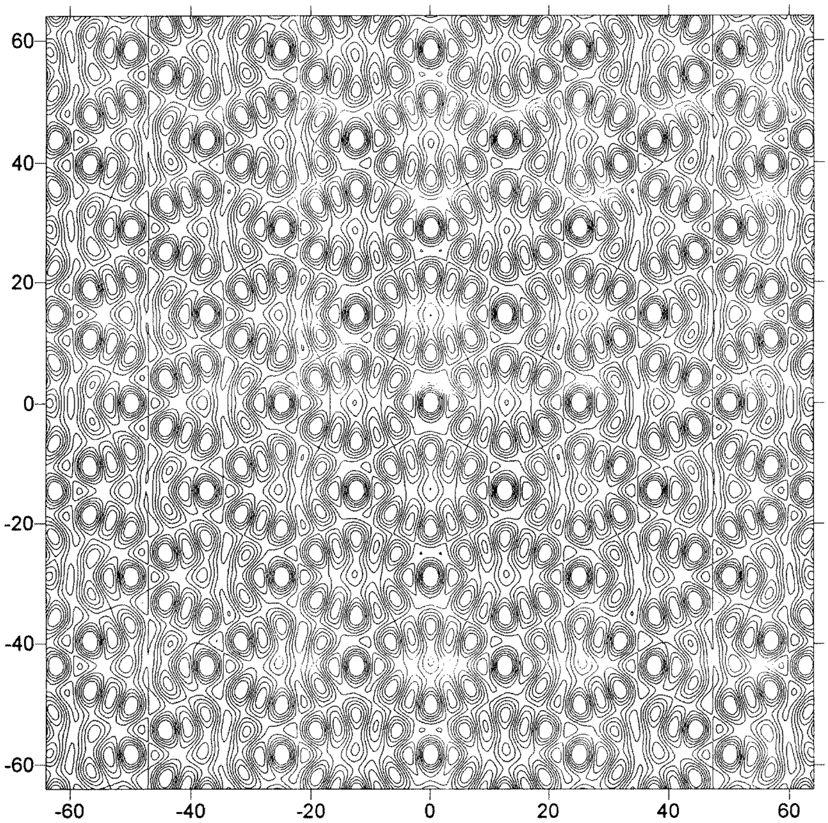


Figure 5. The same as figure 4, for $\alpha = 10^\circ$ (a developed complicated pattern).

5. Conclusions

Thus, we have shown how the method of asymptotical expansions can be modified in order to be effectively applied to a nonlinear scalar field theory (as an example, to the model with quadratic nonlinearity); we have analysed the first iterations of this method. We have shown that at this stage we already obtain QP 2D patterns which include patterns with approximate nontrivial (5-fold) symmetry.

The same approach can be applied to a scalar field theory with *cubic* nonlinearity. Such a model arises in various applications; see, for example, the book [19] and references therein.

In order to construct and describe QP 2D patterns we introduce a 4D space of phase variables and look for strictly 2π-periodic solutions of the associated 4D equation in phase variables. Such solutions are then 'projected' back onto the (x, y)-plane, where their images are QP 2D solutions of the original equation.

The question of the stability of the QP patterns thus obtained lies beyond the scope of our paper. On the general problem on stability of QP solutions we could send readers to the papers of Moser [20] and Bridges [21], and references therein.

As we remarked in the introduction, further development in the study of QP patterns in scalar field models may be related to a proper analysis of QP solutions of the equation

$$\Delta u + \Lambda u = 0 \tag{63}$$

(Δ is the Laplace operator in $\mathcal{R}^2$). This would allow us to construct a proper QP basis as a set of QP eigenfunctions of the eigenvalue problem for this operator in the class of bounded functions in $\mathcal{R}^2$. Such bases would be useful for an expansion of QP solutions of nonlinear equations.

We have taken first steps in this direction. In particular, we have found that the main renormalized mode in the case of 2D resonance corresponds (under certain conditions) to one of the QP modes provided by the eigenvalue problem for the Laplace operator in $\mathcal{R}^2$; moreover, the pattern of this QP mode includes patches with the same nontrivial symmetry as that main mode. This allows us to suppose that for certain models of nonlinear field (solid state) the nontrivial topology of small-amplitude patterns is essentially determined by the pattern topology of QP modes of the *linear* eigenvalue problem for the Laplace operator.

Acknowledgments

The authors are grateful to Professors V I Yudovich, E E Shnol' and V V Men'shenin for fruitful discussions of our results. This work was partly supported by The Russian Basic Research Foundation, Grant No 99-01-00342.

Appendix. Interplay between approximate symmetries of 5th and 12th order in the 2D resonance QP pattern

Consider the function (52) and its local exact and approximate symmetries in detail. This function can be written as a sum of exponents:

$$u = \sum_{n=1}^{12} C_n \exp(\mathrm{i}[k_x^{(n)} x + k_y^{(n)} y]) \qquad C_n = \tfrac{1}{4} \tag{64}$$

where $(k_x^{(n)}, k_y^{(n)})$ are coordinates of 12 points in the plane of wave numbers; these points lie on the unit circle and are equally spaced:

$$k_x^{(n)} = \cos\left[\frac{2\pi}{24}(2n+1)\right] \qquad k_y^{(n)} = \sin\left[\frac{2\pi}{24}(2n+1)\right]. \tag{65}$$

If the original pattern is rotated by the angle β around the point $(0, 0)$ in the plane $\{x, y\}$, then its Fourier image (the set of 12 points) is rotated by the same angle around the point $(0, 0)$ in the plane $\{k_x, k_y\}$, and *vice versa*. Since the set (65) maps onto itself if rotated by the angle $\beta = 2\pi/12$ and all the points have equal 'weights' C_n, the pattern possesses the *exact* symmetry of 12th order relative the origin $(0, 0)$.

The quasiperiodicity of function (52) implies that this pattern must reproduce itself with various accuracy in the vicinity of infinitely many points in the plane $\{x, y\}$. This phenomenon is illustrated in figures 2(*a*)–(*c*). On the other hand, as is clearly seen in these figures, there exist many domains with apparent symmetry of the 5th order.

Let us try to analyse the 'interference' between the 'far' symmetry of 12th and 'near' symmetry of 5th order (figures 1 and 2 bear witness that these two symmetries prevail in the pattern).

Shift the origin of coordinates to an arbitrary point with polar coordinates (r, ϕ). In the plane of wave numbers this procedure changes weights $C_1, \ldots, C_{12}$; they become complex:

$$C_n \Longrightarrow \tilde{C}_n = C_n \exp\left(\mathrm{i}r\cos\left[\frac{2\pi}{24}(2n+1) - \phi\right]\right) \tag{66}$$

but the location of each point (65) remains intact. Fix the direction of the shift (the angle ϕ).

At the point (r, ϕ) we introduce local polar coordinates (R, τ); then

$$u(R, \tau) = \frac{1}{4}\sum_{n=1}^{12} \exp\left(\mathrm{i}r\cos\left[\frac{2\pi}{24}(2n+1) - \phi\right]\right)\exp\left(\mathrm{i}R\cos\left[\frac{2\pi}{24}(2n+1) - \tau\right]\right). \tag{67}$$

Let us demand that the pattern u approximately turns into itself being rotated by the angle $\Delta\tau$ around the point (r, ϕ). Such a rotation is accompanied by the rotation of all 12 points in the Fourier plane $\{k_x, k_y\}$ by the same angle. Hence, we can suppose that the following two conditions are essential for a realization of some (local or global) symmetry of the pattern:

(A) the set of 12 equally spaced points on the circle must map onto itself with good accuracy being rotated by angle $\Delta\tau$ around the origin;

(B) the weight of each point must be approximately equal to the weight associated with its ‘image’ under the above rotation.

This means that each of 12 terms in (67) must map onto another term with more or less accuracy:

$$\begin{aligned}&\exp\left(\mathrm{i}r\cos\left[\frac{2\pi}{24}(2n_1+1) - \phi\right]\right)\exp\left(\mathrm{i}R\cos\left[\frac{2\pi}{24}(2n_1+1) - \tau\right]\right)\\ &\qquad\approx \exp\left(\mathrm{i}r\cos\left[\frac{2\pi}{24}(2n_2+1) - \phi\right]\right)\exp\left(\mathrm{i}R\cos\left[\frac{2\pi}{24}(2n_2+1) - \tau - \Delta\tau\right]\right)\\ &\qquad n_2 = n_2(n_1, \Delta\tau) \qquad n_1 = 1, \ldots, 12.\end{aligned} \tag{68}$$

It is natural to relate possible values of $\Delta\tau$ (possible types of symmetries) with the solvability of 12 approximate equations:

$$\begin{aligned}&R\left\{\cos\left[\frac{2\pi}{24}(2n_1+1) - \tau\right] - \cos\left[\frac{2\pi}{24}(2n_2+1) - \tau - \Delta\tau\right]\right\} \approx 0\\ &n_2 = n_2(n_1, \Delta\tau) \qquad n_1 = 1, \ldots, 12\end{aligned} \tag{69}$$

which express condition (A). There exist two cases:

- the expression in the curly brackets vanishes *exactly*; then the value of R is not important, so the symmetry can be global (or, at least, ‘far’);
- this expression vanishes only *approximately*; in this case the left-hand side of (69) can be made small if one requires smallness of R.

Rewrite the left-hand side of (69) in the form

$$-2R\sin\frac{1}{2}\left\{\frac{2\pi}{12}(n_1+n_2+1) - 2\tau - \Delta\tau\right\}\ \sin\frac{1}{2}\left\{\frac{2\pi}{12}(n_1-n_2) + \Delta\tau\right\}. \tag{70}$$

The first case corresponds to the angles $\Delta\tau = \frac{2\pi}{12}M$, $n_2 - n_1 = M$; $M = 1, 2, 3, 4, 6$ (these angles are associated with symmetries of 12th, 6th, 4th, 3rd and 2nd orders).

The second case works if we take $\Delta\tau = 4\pi/5$. Really, compare this rotation with the rotation of the 12 points by the angle $5 \times 2\pi/12$ (i.e., the shift by 5 points along the circle):

$$5\frac{2\pi}{12} - \frac{4\pi}{5} = \frac{\pi}{30} \ll 1.$$

So we can suppose that the case $n_2 - n_1 = 5$ can also be associated with a symmetry of the pattern (52) (obviously, it is the 5th-order symmetry), but this symmetry can be local only.

Now turn to condition (B) (approximate equality of the ‘weights’, which is necessary for a good reproduction of the pattern upon rotation by the angle $\Delta\tau$). As follows from (68), this condition can be written in the form

$$\begin{aligned}&\frac{r}{2\pi}\left[G(n_1) - G(n_1+M)\right] \approx N(n_1) \in Z\\ &G(m) \equiv \cos\left[\frac{2\pi}{24}(2m+1) - \phi\right] \qquad n_1 = 1, \ldots, 12.\end{aligned} \tag{71}$$

Table A1.

P	Q	N_1	N_2	N_3	r	$x = r/\sqrt{2}$	x_{corr}	x_{num}
2	1	3	4	1	4π	8.88	9.48	9.57
5	3	8	11	3	12π	26.66	26.14	26.15
7	4	11	15	4	16π	35.54	35.71	35.73
19	11	30	41	11	44π	97.74	97.62	97.61
26	15	41	56	15	60π	133.29	133.34	133.34

Solving these 12 equations provides a set of coordinates (r_k, ϕ_k) of the 'approximate symmetry' centres (the order of the symmetry is determined by the value of $n_2 - n_1$, which is found on the first step).

As an example, let us search for symmetry centres that lie on the line $\phi = \pi/4$. Note that if equations (71) are solved for some point $(r^*, \pi/4)$, then these equations are thus solved for all 12 points of kind $(r^*, \phi_m = \frac{2\pi}{24}(2m+1))$.

Let us start with the 5th-order approximate symmetry ($n_2 = n_1 + 5$), for which $G(n_1) - G(n_1+5)$ takes one of six possible values: $\{\pm\frac{\sqrt{3}+1}{2}, \pm\frac{\sqrt{3}+2}{2}, \pm 1/2\}$. Thus, twelve conditions (71) are reduced to three approximate equations for the continuous variable r and three integers (N_1, N_2, N_3):

$$\frac{\sqrt{3}+1}{2} \approx \frac{2\pi}{r} N_1 \qquad \frac{\sqrt{3}+2}{2} \approx \frac{2\pi}{r} N_2 \qquad \frac{1}{2} \approx \frac{2\pi}{r} N_3. \tag{72}$$

The solvability of this system is directly related to an approximation of the irrational number $\sqrt{3}$ by rational ones. Let $\sqrt{3} \approx P/Q$, $P, Q \in Z$. Then the solution of the system (72) has the form

$$N_1 = P + Q \qquad N_2 = P + 2Q \qquad N_3 = Q \qquad r = 4\pi Q. \tag{73}$$

As rational approximations P/Q let us take a sequence of so-called 'convergents of a continued fraction', which are generated when the expansion of $\sqrt{3}$ into a continued fraction

$$\sqrt{3} = 1 + 1/(1 + 1/(2 + 1/(1 + 1/(2 + 1/(1 + 1/(2 + \cdots$$

is cut at subsequent levels:

$$\tfrac{2}{1}, \tfrac{5}{3}, \tfrac{7}{4}, \tfrac{19}{11}, \tfrac{26}{15}, \tfrac{71}{41}, \ldots \Longrightarrow \sqrt{3}. \tag{74}$$

Corresponding values of N_1, N_2, N_3, and r are presented in table A1.

Note that the above rational approximations are not unique; for example, since $\sqrt{3} = 3/\sqrt{3}$, the number $\sqrt{3}$ can be approximated by rational fractions $3Q/P$ with the same P, Q as above.

It follows from (66) that, as r varies, each weight coefficient $\tilde{C}_n$ moves in the complex plane along the circle with constant 'velocity', which depends on n. Looking at the mapping (68) and equation (71) we see that for the values of r presented in the table the phase differences between $\tilde{C}_n$ and $\tilde{C}_{n+5}$ simultaneously (i.e., for all 12 such pairs) become very close to 2π (multiplied by the integers N_1, N_2, or N_3).

Numerical analysis proves that all the points $(r, \phi/4)$ with r taken from above table coincide (with very good accuracy) with centres of patches that are associated with the symmetry of 5th order (compare values of $x = r/\sqrt{2}$ with numerical values of this coordinate x_{num}; see also figures 1 and 2); the larger are P, Q, the better is our evaluation of coordinate r of the corresponding symmetry centre.

In the case of 12th-order symmetry expressions $G(n_1) - G(n_1+1)$ take one of six values: $\{\pm\frac{\sqrt{3}-1}{2}, \pm\frac{\sqrt{3}-2}{2}, \pm\frac{1}{2}\}$. In this case we get the following three approximate equations:

$$\frac{\sqrt{3}-1}{2} \approx \frac{2\pi}{r}N_1 \qquad \frac{2-\sqrt{3}}{2} \approx \frac{2\pi}{r}N_2 \qquad \frac{1}{2} \approx \frac{2\pi}{r}N_3. \tag{75}$$

Substituting the rational fraction P/Q, $P, Q \in \boldsymbol{Z}$ for $\sqrt{3}$, we find

$$N_1 = P - Q \qquad N_2 = 2Q - P \qquad N_3 = Q \qquad r = 4\pi Q. \tag{76}$$

Thus, we have encountered an intriguing fact: the points of the plane (r, ϕ) that were found as probable centres of the approximate 5th-order symmetry are at the same time probable centres of the approximate 12th-order symmetry. Which symmetry 'wins'?

Let us suppose that it is the symmetry for which the weight coefficients $\tilde{C}_n$ are reproduced better upon rotation by angle $2\pi/12 \cdot (n_2 - n_1)$; formally, it is the symmetry that minimizes the average deviation ε of an argument of the weight coefficient divided by 2π from an integer:

$$\varepsilon = \frac{1}{12}\sum_{n_1=1}^{12}\left|\frac{r}{2\pi}(G(n_1) - G(n_2)) - \mathrm{int}\left[\frac{r}{2\pi}(G(n_1) - G(n_2))\right]\right| \tag{77}$$

where int[ ] means taking the nearest integer.

So we should compare the average deviation ε for the 5th- and 12th-order symmetries. Before we proceed, note that the coordinates r of the 'symmetry centres' that were calculated above contain an additional error related with the rational approximations used instead of exact $\sqrt{3}$. To evaluate the deviation ε with proper accuracy, we should try to reduce the deviation taking into account this error. Let $\Delta = \Delta(P, Q)$ be an error in the coordinate r of the symmetry centre: $\Delta = r - 4\pi Q$, and let $\delta = \delta(P, Q)$ be an error of the rational approximation of $\sqrt{3}$: $\delta = \sqrt{3} - \frac{P}{Q}$.

Calculating the deviation ε for each equation in the triplets (72) and (75) in linear approximation w.r.t. Δ and δ and averaging, we find that the minimum of the average deviation for the first case equals

$$\min_{\Delta}\varepsilon^{(5)}_{\mathrm{average}} = \frac{2}{3}\delta\frac{Q^2}{P+2Q} \tag{78}$$

and is attained at $\Delta_{\min} = -4\pi\delta\frac{Q^2}{P+2Q}$.

So, we have simultaneously calculated a correction for the coordinate of the 5th-approximate symmetry centres. It is in good accordance with numerical data: compare the corrected values $x_{\mathrm{corr}} = x + \Delta_{\min}/\sqrt{2}$ with the numerical values x_{num} in table A1.

In the case of 12th-order symmetry we find that

$$\min_{\Delta}\varepsilon^{(12)}_{\mathrm{average}} = \tfrac{2}{3}Q\delta \qquad \Delta_{\min} = 0 \tag{79}$$

so any additional shift in r does not allow one to reduce the deviation.

Thus, for the ratio of the deviations for the symmetries of 5th and 12th order we find:

$$\min_{\Delta}\varepsilon^{(5)}_{\mathrm{average}} \Big/ \min_{\Delta}\varepsilon^{(12)}_{\mathrm{average}} \sim \frac{Q}{P+2Q} \tag{80}$$

so the deviation for the shift $n_2 - n_1 = 5$ is always several times less than for $n_2 - n_1 = 1$. Say, for $Q = 3$, $P = 5$ this ratio equals $\frac{3}{11}$; for large numbers P, Q it tends to $1/(2+\sqrt{3}) \sim \frac{1}{4}$. This evaluation was confirmed by our numerical analysis.

Hence, we have shown that, for not too large R, the conditions for the 5th-order symmetry to appear are always more 'favourable' than those for the 12th-order symmetry. The numerical

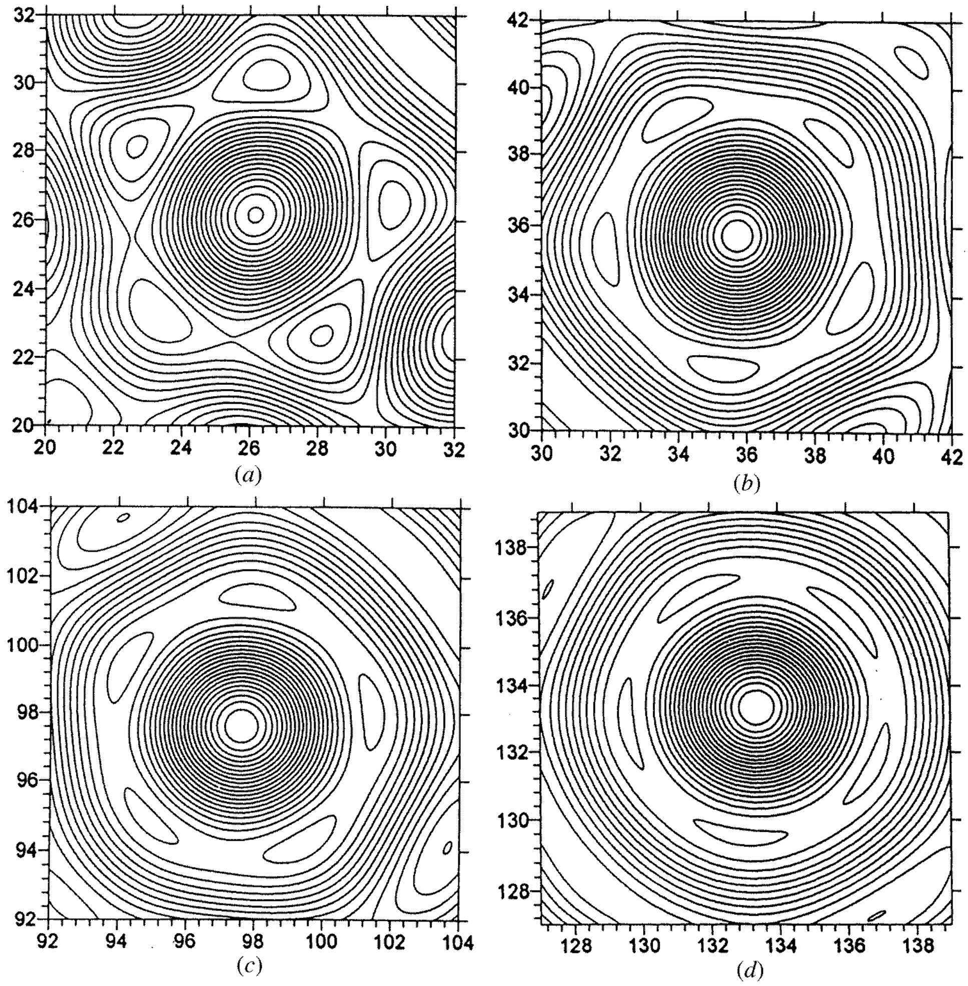


Figure A1. Local patterns with approximate 5-fold symmetry. Graphs (*c*) and (*d*) are the central parts of the patterns with good 'far' 12-fold symmetry presented in figures 2(*b*)–(*c*).

simulation proves that at the points with small P, Q, where the deviation ε for the 12th-order symmetry is still too large, its value for the 5th-order symmetry is already quite small; and, really, inside domains with $R < 4$ around such points we see apparent local patterns with this symmetry (in spite of the fact that condition (A) is satisfied for angle $\Delta\tau = 4\pi/5$ only approximately). Such domains are the main elements generating the global pattern of our function. At the points with P and Q large enough (actually, already for $P = 7, Q = 4$) the deviation for the 12th-order symmetry also becomes acceptably small. In such cases we observe much larger domains with the 12th-order symmetry; and the better is the rational approximation of $\sqrt{3}$, the more similar is the local pattern to the pattern around the origin (figures 1 and 2) with exact 12th-order symmetry.

But the ratio of the deviations is *always* about $\frac{1}{4}$, so at the vicinity of the centre of such a pattern the 5th-order symmetry nevertheless prevails and suppresses the 12th-order symmetry! As P and Q grow, the 'near' 5th-order symmetry around the centre of the local pattern with 'far' 12th-order symmetry becomes more and more exact; this phenomenon is clearly seen in figures A1(*a*)–(*d*).

References

[1] Gledzer E B, Dolzhansky F V and Obukhov A M 1981 *Systems of Hydrodynamic Type and their Application* (Moscow: Nauka)
[2] Brambilla M, Battipede F, Lugiato L A *et al* 1991 *Phys. Rev.* A **43** 5090
[3] Papoff F, D'Alessandro G, Oppo G-L and Firth W J 1993 *Phys. Rev.* A **48** 634
[4] Shechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951
[5] Nelson D R and Halperin B I 1985 *Science* **229** 235
[6] Zaslavsky G M, Sagdeev R Z, Usikov D A and Chernikov A A 1988 *Usp. Fiz. Nauk* **156** 193
[7] Steinhardt P and Ostlund S (ed) 1987 *The Physics of Quasicrystals* (Singapore: World Scientific)
[8] Christiansen B, Altrom P and Levinsen M T 1992 *Phys. Rev. Lett.* **68** 2157
[9] Edwards W S and Fauve S 1993 *Phys. Rev.* E **47** 788
[10] Arnol'd V I 1988 *Physica* D **33** 21
[11] Arnol'd V I 1989 *Huygens and Barrow, Newton and Hooke* (Moscow: Nauka)
[12] Penrose R 1974 *Bull. Inst. Math. Appl.* **10** 266
Penrose R 1978 *Eureka* **39** 16
[13] Guy R K 1976 *Bull. Lond. Math. Soc.* **8** 9
[14] Grünbaum B and Shepard G C 1986 *Tilings and Patterns* (San Francisco: Freeman)
[15] Batchelor G K 1970 *An Introduction to Fluid Dynamics* (Cambridge: Cambridge University Press)
[16] Eleonsky V M, Korolev V G and Kulagin N E Regular and chaotic dynamics *Proc. 28th Int. Workshop 'Kourovka-2000' (Chelyabinsk, 28 February–4 March 2000)* (in Russian) submitted
[17] Eleonsky V M and Silin V P 1969 *Zh. Eksp. Teor. Fiz.* **56** 574
Eleonsky V M and Silin V P 1969 *Zh. Eksp. Teor. Fiz.* **57** 478
[18] Nayfeh A H 1973 *Perturbation Methods* (New York: Wiley)
[19] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)
[20] Moser J 1989 *Bol. Soc. Bras. Mat.* **20** 29
[21] Bridges T J 1998 *Nonlinearity* **11** 467